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Pomeron with a running coupling constant in the nucleus

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Abstract. The running coupling is introduced into the equation for propagation of the pomeron in the nucleus via the bootstrap relation. The resulting equation coincides with the one obtained in the color dipole formalism by summing contributions from quark–antiquark loops, with a general choice of the regularization scheme.

1 Introduction

Lately, renewed interest has been shown in the introduction of a running coupling into the BFKL dynamics. Summation of contributions from the quark–antiquark loops to the evolution of the gluon density has been used to restore the full dependence of the coupling on the running scale in the color dipole approach [1–3]. It turned out that the obtained kernel for the linear BFKL equation essentially coincides with the one that we found many years ago by imposing the bootstrap condition necessary for the fulfilment of unitarity [4, 5]. In this paper we draw attention to the fact that the bootstrap condition in fact allows one to derive also the structure of the triple-pomeron interaction and thus the form of the non-linear BFKL equation describing propagation of the pomeron in the nucleus. This is a simple consequence of the possibility to express the basic splitting kernel from two to three and four gluons ('the Bartels vertex' [6]) via the basic BFKL interaction. The relation between the splitting vertex and reggeized gluon interaction and the bootstrap are the necessary ingredients of the pomeron interaction in the reggeized gluon approach. Indeed they allow one to present all contributions to the four-gluon amplitude in the standard form of a pomeron splitting into two pomerons. Thus preserving these two relations seems to be essential for the construction of a pomeron interaction with the running coupling.

As in our earlier papers, we have to stress from the start that the introduction of the running coupling into the BFKL formalism cannot be made rigorously and uniquely. The formalism admits transverse momenta of any magnitude, including very small ones, at which the concept of the gluon and its coupling looses any meaning. The introduced running coupling has to be artificially continued to small momentum values, where it is completely undetermined. The only information one can get from the running of the coupling refers to the region of high momenta. It remains to be seen in what measure this information depends on the low momentum behavior, which cannot be fixed theoretically in any reliable manner. It is known, however, that the non-linear BFKL equation, unlike the linear one, is not very sensitive to the infrared region. So hopefully this equation has a better chance to produce results weakly dependent on the assumed infrared behavior of the running coupling.

Also the method uses the bootstrap equations and the triple-pomeron interaction in the lowest order of the coupling. As a result, one can only hope to establish the leading order behavior in the running coupling. For the linear evolution, in the next-to-leading order, it has been explicitly found that the bootstrap method correctly reproduces the part of the kernel responsible for the running of the coupling but not the rest piece [7–9]. One expects the same to be true also for the non-linear evolution. Finally, it is to be noted that the approximation of large N_c is used for the interaction. For finite N_c the theory ceases to be that of interacting pomerons but rather becomes one of interacting reggeized gluons, which leads to enormous complications for the non-linear evolution.

The paper is organized as follows. The first section is dedicated to the derivation of the triple-pomeron vertex with the running coupling introduced via the bootstrap relation. After recalling this method to introduce the running coupling, we go along the same steps as in [10, 11], where the vertex was derived in the limit $N_c \to \infty$ for the fixed coupling. In Sect. 3 we construct the full amplitude with a single triple-pomeron interaction, coupling the vertex with three pomerons. This result enables us to build the equation for the pomeron in the large nucleus in Sect. 4. Finally in Sect. 5 we compare our results with those obtained within the color dipole approach.

Having in mind the application of the formalism to the large nucleus as a target, we limit ourselves to the case of propagation of pomerons with zero total momenta. However, a generalization to non-zero total momenta is straightforward.

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2 Triple-pomeron vertex

2.1 Generalities

As mentioned in the Introduction, this paper follows the idea of introducing a running coupling via the bootstrap [4, 5]. Derivation of the triple-pomeron vertex in the limit $N_c \to \infty$ then goes as presented in [10, 11] for the fixed coupling case.

Basic formulas for the introduction of a running coupling via the bootstrap condition consist in expressing both the gluon trajectory ω and the intergluon interaction in the vacuum channel K in terms of a single function $\eta(q)$ of the gluon momentum, which then can be chosen to conform to the high-momentum behavior of the gluon density with a running coupling. Explicitly,

$$\omega(q) = -\frac{1}{2}N_c \int \frac{\mathrm{d}^2 q_1}{(2\pi)^2} \frac{\eta(q)}{\eta(q_1)\eta(q_2)}\,, \tag{1}$$

$$K\,(q_1, q_2|q_1' q_2') = N_c \Bigg[\left(\frac{\eta(q_1)}{\eta(q_1')} + \frac{\eta(q_2)}{\eta(q_2')} \right) \frac{1}{\eta(q_1 - q_1')} - \frac{\eta(q_1+q_2)}{\eta(q_1')\eta(q_2')} \Bigg]. \tag{2}$$

In these definitions it is assumed that $q_1 + q_2 = q_1' + q_2' = q$. For arbitrary $\eta(q)$ the following bootstrap relation is satisfied:

$$\frac{1}{2}\int \frac{\mathrm{d}^2 q_1'}{(2\pi)^2} K(q, q_1, q_1') = \omega(q) - \omega(q_1) - \omega(q_2)\,. \tag{3}$$

The fixed coupling corresponds to the choice

$$\eta(q) = \frac{2\pi}{g^2} q^2\,. \tag{4}$$

Then one finds the standard expression for the trajectory $\omega(q)$ and

$$K(q, q_1, q_1') = \frac{g^2 N_c}{2\pi} V(q, q_1, q_1')\,,$$

where V is the standard BFKL interaction. Note that the extra 2π in the denominator corresponds to the BFKL weight $1/(2\pi)^3$ in the momentum integration, which we prefer to take standardly as $1/(2\pi)^2$.

From the high-momentum behavior of the gluon distribution with a running coupling one finds

$$\eta(q) = \frac{1}{2\pi} b q^2 \ln \frac{q^2}{\Lambda^2}\,, \quad q^2 \gg \Lambda^2\,, \tag{5}$$

where Λ is the standard QCD parameter, and

$$b = \frac{1}{12}(11 N_c - 2N_f)\,. \tag{6}$$

As to the behavior of $\eta(q)$ at small momenta, we shall assume

$$\eta(0) = 0\,, \tag{7}$$

which guarantees that the gluon trajectory $\omega(q)$ passes through zero at $q = 0$ in accordance with the gluon properties. The asymptotic equation (5) and the condition (7) are the only properties of $\eta(q)$ that follow from the theoretical reasoning. A concrete form of $\eta(q)$ interpolating between (7) and (5) may be chosen differently. One hopes that the following physical results will not strongly depend on the choice.

Our old derivation in [10] of the triple-pomeron vertex was actually based on the property (7), obviously valid for (4), the bootstrap relation and the relation between the Bartels transition vertex $K_{2\to 3}$ and the intergluon BFKL interaction V ((12) in [10])

$$K_{2\to 3}\,(1, 2, 3|1', 3') = V\,(2, 3|1' - 1, 3') - V\,(12, 3|1'3')\,. \tag{8}$$

Here and in the following we frequently denote the gluon momenta just by numbers: $1 \equiv q_1$, $1' \equiv q_1'$ etc. Also we use $12 \equiv q_1 + q_2$. All the rest of the conclusions were obtained from these three relations in a purely algebraic manner.

Our idea is that if we *define* the transition vertex by a similar relation in terms of the new intergluon vertex K, (2), then the whole derivation will remain valid also for arbitrary $\eta(q)$ satisfying (7) and thus for a running coupling, provided $\eta(q)$ is chosen appropriately. In the next subsection we briefly recapitulate the successive stages of the derivation for arbitrary $\eta(q)$ with $\eta(0) = 0$.

2.2 Leading order in $N_c \to \infty$

The changes necessary to pass to arbitrary $\eta(q)$ are minimal. Obviously in our old formulas we have to drop the coupling constant factors g, since now g is provided by $1/\eta$. We also drop factors N_c in the interaction because we prefer to include it into the kernel K. With these comments, the two-gluon equation becomes

$$S_{20} D_2 = D_{20} + K_{12} D_2\,, \tag{9}$$

where

$$S_{20} = j - 1 - \omega(1) - \omega(2)\,; \tag{10}$$

j is the angular momentum variable,

$$D_{20} = N_c\,(f(0, q) - f(q_1, q_2)) \tag{11}$$

and $f(q_1, q_2)$ is just the quark–antiquark loop with, say, the gluon with momentum q_1 coupled to the quark and the other one, with momentum q_2, coupled to the antiquark. With the running coupling D_{20} does not change form, but both ω in (10) and K in (9) are now given by (1) and (2), respectively.

The three-gluon system exists in two color states, which differ in the ordering of the three gluons along the loop, 123 and 213. For each order the equation is

$$S_{30} D_3 = D_{30} + D_{2\to 3} + \frac{1}{2}(K_{12} + K_{23} + K_{31}) D_3\,, \tag{12}$$

where now

$$S_{30} = j - 1 - \sum_{j=1}^{3} \omega(j) \tag{13}$$

and

$$D_{30}^{(123)} = -D_{30}^{(213)} = \sqrt{\frac{N_c}{8}} \left(D_{20}(2) - D_{20}(1) - D_{20}(3)\right). \tag{14}$$

The new element is the term $D_{2\to3}$, which corresponds to transitions of the initial two-gluon system into the final three-gluon system. This transition is accomplished by the $2 \to 3$ vertex W, which, as explained, we define via the new intergluon interaction by a relation similar to (8):

$$W\left(1,2,3|1',3'\right) = K\left(2,3|1'-1,3'\right) - K\left(12,3|1'3'\right). \tag{15}$$

Explicitly, in terms of η,

$$\frac{1}{N_c} W(1,2,3|1',3') = \frac{\eta(2)}{\eta(1-1')\eta(3-3')} - \frac{\eta(23)}{\eta(1-1')\eta(3')} - \frac{\eta(12)}{\eta(1')\eta(3-3')} + \frac{\eta(123)}{\eta(1')\eta(3')}. \tag{16}$$

Note that, as with a fixed coupling constant,

$$W\left(1,2,3|1',3'\right) = W\left(3,2,1|3',1'\right). \tag{17}$$

We find in full analogy with [10]

$$D_{2\to3} = \sqrt{\frac{N_c}{8}} W\left(1,2,3|1',3'\right) \otimes D_2\left(1',3'\right) \equiv \sqrt{\frac{N_c}{8}} W(1,2,3), \tag{18}$$

where $\otimes$ means integration over the intermediate gluon momentum with weight $1/(2\pi)^2$.

The next step is to show that (12) for the three-gluon system is solved by the reggeized zero term ansatz:

$$D_3^{(123)} = -D_3^{(213)} = \sqrt{\frac{N_c}{8}} \left(D_2(2) - D_2(1) - D_2(3)\right), \tag{19}$$

that is, by (14), in which the loops are substituted by the full solutions of the two-gluon equation (9). The proof is wholly based on the bootstrap and (15) and literally repeats the corresponding proof in [10].

Passing to the four-gluon system, in the limit $N_c \to \infty$ we find two configurations differing by the order of gluons along the quark–antiquark loop: 1234 and 2134. The equation governing the four-gluon system is

$$S_{40}D_4 = D_{40} + D_{2\to4} + D_{3\to4} + \frac{1}{2}(K_{12} + K_{23} + K_{34} + K_{41})D_4, \tag{20}$$

where

$$S_{40} = j - 1 - \sum_{j=1}^{4} \omega(j). \tag{21}$$

The inhomogeneous terms are

$$D_{40}^{(1234)} = \frac{1}{4} N_c \left(D_{20}(1) + D_{20}(4) - D_{20}(14)\right), \tag{22}$$

$$D_{40}^{(2134)} = \frac{1}{4} N_c \left(D_{20}(2) + D_{20}(3) - D_{20}(12) - D_{20}(13)\right), \tag{23}$$

$$D_{2\to4}^{(1234)} = -\frac{1}{4} N_c W(1,23,4), \quad D_{2\to4}^{(2134)} = 0 \tag{24}$$

(the definition of $W(1,2,3)$ is given by the second equality in (18)),

$$D_{3\to4}^{(1234)} = \sqrt{\frac{N_c}{8}} \Big(W(2,3,4|2',4') \otimes D_3^{(124)}(1,2',4') + W(1,2,3|1',3') \otimes D_3^{(134)}(1',3',4)\Big), \tag{25}$$

and

$$D_{3\to4}^{(2134)} = -\sqrt{\frac{N_c}{8}} \Big(W(1,2,4|1',4') \otimes D_3^{(134)}(1',3,4') + W(1,3,4|1',4') \otimes D_3^{(124)}(1',2,4')\Big). \tag{26}$$

Repeating the corresponding derivation in [10] we demonstrate that in the limit $N_c \to \infty$ the solution of the four-gluon equation is again given by the reggeized zero order terms:

$$D_4^{(1234)} = \frac{1}{4} N_c (D_2(1) + D_2(4) - D_2(14)) \tag{27}$$

and

$$D_4^{(2134)} = \frac{1}{4} N_c (D_2(2) + D_2(3) - D_2(12) - D_2(13)). \tag{28}$$

The proof is purely algebraic and is wholly based on the bootstrap and (15) and (7) valid for any choice of the function $\eta(q)$ with $\eta(0) = 0$.

2.3 The triple-pomeron configuration

The next step is to study the next-to-leading configuration in $N_c \to \infty$ corresponding to the triple-pomeron interaction. Again the derivation practically literally repeats our old one for a fixed coupling constant. The governing four-gluon equation is similar to (20)

$$S_{40}D_4 = D_{40} + D_{2\to4} + D_{3\to4} + D_{4\to4} + (K_{12} + K_{34})D_4. \tag{29}$$

The formal difference is in the absence of interaction between the two final pomerons, which are assumed to be

made of gluon pairs (1,2) and (3,4), doubling of the rest interactions acting in the vacuum color channels and in the appearance of the term $D_{4\to4}$, which describes transitions from the leading color configuration to the subleading one corresponding to two pomerons.

The four inhomogeneous terms are

$$D_{40} = \frac{1}{2}\left(\sum_{j=1}^{4} D_{20}(j) - \sum_{j=2}^{4} D_{20}(1j)\right), \tag{30}$$

$$D_{2\to4} = -W(1,23,4)\,, \tag{31}$$

$$\begin{aligned} D_{3\to4} = \sqrt{\frac{2}{N_c}}\Big(&W(1,2,3|1',3')\otimes D_3^{(134)}(1',3',4)\\ &- W(1,2,4|1',4')\otimes D_3^{(134)}(1',3,4')\\ &+ W(2,3,4|2',4')\otimes D_3^{(124)}(1,2',4)\\ &- W(1,3,4|1',4')\otimes D_3^{(124)}(1',2,4)\Big) \end{aligned} \tag{32}$$

and

$$D_{4\to4} = \frac{1}{N_c}(K_{23}+K_{14}-K_{13}-K_{24})\left(D_4^{(1234)} - D_4^{(2134)}\right), \tag{33}$$

where the D_4 on the right-hand side are given by (27) and (28). As with a fixed coupling constant, all terms except D_{40} can be presented as a result of the action of a certain operator Z on the two-gluon state, so that (29) can be rewritten as

$$S_{40}D_4 = D_{40} + Z\otimes D_2 + (K_{12}+K_{34})D_4\,. \tag{34}$$

The explicit form of this operator can be expressed in terms of the function

$$\begin{aligned} G(1,2,3) = &-W(1,2,3) - D_2(1,23)(\omega(2)-\omega(23))\\ &- D_2(12,3)(\omega(2)-\omega(12))\,. \end{aligned} \tag{35}$$

Then one finds

$$\begin{aligned} Z\otimes D_2 = \frac{1}{2}&(2G(1,34,2)+2G(3,12,4)+G(1,23,4)\\ &+G(1,24,3)+G(2,13,4)+G(2,14,3)\\ &-G(1,3,24)-G(1,4,23)-G(2,3,14)\\ &-G(2,4,13)-G(3,2,14)\\ &-G(3,1,24)-G(4,2,13)-G(4,1,23)\\ &+G(23,0,14)+G(13,0,24))\,. \end{aligned} \tag{36}$$

This formula is identical to the old one in [10], but with new expressions for both W and ω in terms of the function η.

2.4 The triple-pomeron vertex

Corresponding to the two inhomogeneous terms in (34) its solution is split in two terms, the double pomeron exchange term, generated by D_{40}, and the triple-pomeron interaction term $Z\otimes D_2$. However, one can simplify the solution transferring the part of the double pomeron exchange term leading in the high-energy limit into the triple interaction part [11, 12]. This is achieved by separating from the total solution the reggeized D_{40} term:

$$D_4 = D_{40}(D_{20}\to D_2) + D_4^I\,. \tag{37}$$

The irreducible part D_4^I proves to be a pure triple-pomeron interaction, which satisfies

$$D_4^I = Y\otimes D_2 + (K_{12}+K_{34})D_4^I\,. \tag{38}$$

The derivation again uses only the bootstrap, the relation (15) between W and K and the property (7). The explicit form for the new triple-pomeron vertex Y turns out to be

$$\begin{aligned} Y\otimes D_2 = \frac{1}{2}&G(1,23,4)+G(1,24,3)+G(2,13,4)\\ &+G(2,14,3)+G(12,0,34)\\ &-G(1,2,34)-G(2,1,34)-G(3,4,12)\\ &-G(4,3,12))\,. \end{aligned} \tag{39}$$

For a fixed coupling constant this expression was found long ago in [12]. In our approach it remains true also for an arbitrary function $\eta(q)$ and thus for a running coupling introduced by means of this function.

3 Coupling to pomerons

3.1 Momentum space approach

In the momentum space coupling the vertex Y to two outgoing pomerons is straightforward. The two pomerons are described by a product $P(1,2)P(3,4)$, and all one has to do is to integrate this product with $Y\otimes D_2$ over the gluon momenta 1, 2, 3 and 4 with 12 and 34 fixed and $1234=0$ from momentum conservation.

As with a fixed coupling constant, there are certain properties of the wave function and the vertex that simplify the resulting expression. First we expect that the pomeron wave function in the coordinate space $P(r_1,r_2)$ vanishes if the two gluons are located at the same spatial point: $P(r,r)=0$. This property is well known for the BFKL pomeron with a fixed coupling constant and is related to the behavior of (4) at $q=0$. With a running coupling this behavior does not change, so we expect that the coordinate wave function will continue to vanish at $r_1=r_2$. As a result the last five terms in the sum (39) will give no contribution, since they depend only on the sum of the momenta in one of the pomerons and so put the two gluons at the same spatial point in it. If the mentioned property of the pomeron wave function is violated with the introduction of a running coupling, we still can drop the five last terms once we restrict ourselves to the case when the two pomerons are taken at zero total momentum, $12=34=0$, which is the only case relevant for the pomeron propagation through the nucleus. Indeed direct inspection shows that

$$G(q_1,q_2,q_3) = 0\,, \quad \text{if} \quad q_1=0\,, \quad \text{or} \quad q_3=0\,. \tag{40}$$

Second, due to the symmetry of the pomeron wave function, each of the four remaining terms in (39) gives the same contribution. So, as for the fixed coupling, the triple-pomeron vertex effectively reduces to

$$Y \otimes D_2 = 2G(1,23,4)\,. \tag{41}$$

Coupling this vertex to two forward pomerons and using (35) we get an expression for the triple-pomeron interaction amplitude T in terms of the function η (suppressing the dependence on rapidities),

$$\begin{aligned} T = 2N_c \int \frac{\mathrm{d}^2 q_1\,\mathrm{d}^2 q_4\,\mathrm{d}^2 q_1'}{(2\pi)^6} P(1,2)P(3,4)\Big\{ &- D_2(1',-1') \\ &\times \left(\frac{\eta(14)}{\eta(1-1')\eta(4-4')} - \frac{\eta(1)}{\eta(1-1')\eta(4')} - \frac{\eta(4)}{\eta(1')\eta(4-4')}\right) \\ &+ \frac{1}{2} D_2(1,-1)\left(\frac{\eta(14)}{\eta(1')\eta(14-1')} - \frac{\eta(1)}{\eta(1')\eta(1-1')}\right) \\ &+ \frac{1}{2} D_2(4,-4)\left(\frac{\eta(14)}{\eta(1')\eta(14-1')} - \frac{\eta(4)}{\eta(1')\eta(4-1')}\right)\Big\}\,, \end{aligned} \tag{42}$$

with $12 = 34 = 1'4' = 0$. In this formula actually both D_2 and P are the pomeron wave functions for the incoming and outgoing pomerons, respectively. The function D_2 is the amputated wave function related to P by the relation

$$P(1,-1) = \frac{1}{\eta^2(1)} D_2(1,-1)\,. \tag{43}$$

Expression (42) is rather cumbersome. A simpler expression is obtained in the coordinate representation, which will presently be derived.

3.2 Coordinate space approach

We represent

$$\begin{aligned} &(2\pi)^2\delta^2(q_{12}-q_1-q_2)P(q_1,q_2) \\ &= \int \mathrm{d}^2 r_1\,\mathrm{d}^2 r_2 P_{q_{12}}(r_1,r_2)\,\mathrm{e}^{\mathrm{i}q_1 r_1 + \mathrm{i}q_2 r_2}\,, \end{aligned} \tag{44}$$

where $P_{q_{12}}(r_1,r_2)$ is the coordinate wave function of the pomeron with the total momentum q_{12}. Similarly we represent the second pomeron via $P_{q_{34}}(r_3,r_4)$. Finally,

$$\begin{aligned} &G(q_1,q_2+q_3,q_4) \\ &= \int \mathrm{d}^2 r_1\,\mathrm{d}^2 r_2\,\mathrm{d}^2 r_3\,\mathrm{e}^{-\mathrm{i}q_1 r_1 - \mathrm{i}(q_2+q_3)r_2 - \mathrm{i}q_4 r_3} G(r_1,r_2,r_3)\,. \end{aligned} \tag{45}$$

Then integration over $q_1,\ldots,q_4$ gives

$$\begin{aligned} &(2\pi)^2\delta^2(q_{12}+q_{34})T \\ &= 2\int \mathrm{d}r^2 r_1\,\mathrm{d}^2 r_2\,\mathrm{d}^2 r_3 P_{q_{12}}(r_1,r_2)P_{q_{34}}(r_3,r_2)G(r_1,r_2,r_3)\,. \end{aligned} \tag{46}$$

We have

$$P_{q_{12}}(r_1,r_2) = \mathrm{e}^{\frac{1}{2}\mathrm{i}q_{12}(r_1+r_2)} P(r_{12})\,, \tag{47}$$

where $r_{12} = r_1 - r_2$. So (46) becomes

$$\begin{aligned} &(2\pi)^2\delta^2(q_{12}+q_{34})T \\ &= 2\int \mathrm{d}^2 r_1\,\mathrm{d}^2 r_2\,\mathrm{d}^2 r_3\,\mathrm{e}^{\frac{1}{2}\mathrm{i}[q_{12}r_1 + (q_{12}+q_{34})r_2 + q_{34}r_3]} \\ &\quad \times P(r_{12})P(r_{32})G(r_1,r_2,r_3)\,. \end{aligned} \tag{48}$$

At this stage we note that if we drop $P(r_{12})$ from the integrand we get

$$\begin{aligned} &\int \mathrm{d}r^2 r_1\,\mathrm{d}^2 r_2\,\mathrm{d}^2 r_3\,\mathrm{e}^{\frac{1}{2}\mathrm{i}[q_{12}r_1 + (q_{12}+q_{34})r_2 + q_{34}r_3]} \\ &\quad \times P(r_{32})G(r_1,r_2,r_3) \\ &= \int \mathrm{d}^2 r_2\,\mathrm{d}^2 r_3\,\mathrm{e}^{\frac{1}{2}\mathrm{i}[(q_{12}+q_{34})r_2 + q_{34}r_3]} P(r_{32})G(q_{12},r_2,r_3)\,. \end{aligned} \tag{49}$$

If $q_{12} = 0$, then this expression vanishes due to the property (40). As a result one can substitute in (48)

$$P(r_{12}) \to P(r_{12}) - P(0)\,, \quad P(r_{32}) \to P(r_{32}) - P(0)\,. \tag{50}$$

So whether $P(0)$ is equal to zero or not, for the forward case one can always make it equal to zero by the substitution (50). So in the following we assume $P(0) = 0$.

As follows from translational invariance, for the overall zero total momentum $G(r_1,r_2,r_3) = G(r_{12},r_{32})$. So taking as integration variables r_2, r_{12} and r_{32} we finally obtain

$$T = 2\int \mathrm{d}^2 r_{12}\,\mathrm{d}^2 r_{32} P(r_{12})P(r_{32})G(r_1,r_2,r_3)\,. \tag{51}$$

Now we have to calculate $G(r_1,r_2,r_3)$. Due to $P(0) = 0$ we may drop all terms containing $\delta^2(r_{12})$ and/or $\delta^2(r_{23})$. The total contribution consists of two parts, the first one coming from the term $-W$ in (35) and the second one from the rest. As for the fixed coupling constant case [11] in the first part only the first term in (16) gives a contribution that does not contain $\delta^2(r_{12})$, nor $\delta^2(r_{23})$, nor both. Direct calculation gives for this contribution

$$\begin{aligned} G_1(r_1,r_2,r_3) = &-N_c D_2(r_1,r_3) \\ &\times \int \mathrm{d}^2\rho\tilde{\eta}(\rho)\xi(r_{12}-\rho)\xi(r_{32}-\rho)\,, \end{aligned} \tag{52}$$

where $\tilde{\eta}(r)$ is the Fourier transform of $\eta(q)$, and $\xi(r)$ is the Fourier transform of $1/\eta(q)$. From the four terms in the second part the contribution that does not contain $\delta^2(r_{12})$, nor $\delta^2(r_{23})$, nor both, comes from the first and third terms. Its calculation gives

$$\begin{aligned} G_2(r_1,r_2,r_3) = &\frac{1}{2} N_c D_2(r_1,r_3) \\ &\times \int \mathrm{d}^2\rho\tilde{\eta}(\rho)\left(\xi^2(r_{12}-\rho) + \xi^2(r_{32}-\rho)\right). \end{aligned} \tag{53}$$

Summing we get the final expression

$$G(r_1,r_2,r_3)=\frac{1}{2}N_cD_2(r_{13})F(r_{12},r_{32})\,, \qquad (54)$$

where

$$F(r_1,r_2)=\int \mathrm{d}^2\rho\tilde{\eta}(\rho)\left(\xi(r_1-\rho)-\xi(r_2-\rho)\right)^2\,, \quad (55)$$

and we also used that for the forward pomeron $D_2(r_1,r_3)=D_2(r_{13})$. The triple-pomeron contribution to the amplitude is then obtained as

$$T(Y)=N_c\int_0^Y \mathrm{d}y\int \mathrm{d}^2r_{12}\,\mathrm{d}^2r_{32}F(r_{12},r_{32})P(Y-y,r_{12}) \times P(Y-y,r_{32})D_2(y,r_{13})\,, \qquad (56)$$

where we restored the y-dependence, suppressed up to now.

Note that for a fixed coupling constant we have (see the appendix and also [11])

$$G^{\mathrm{fix}}(r_1,r_2,r_3)=-\frac{g^2N_c}{8\pi^3}\frac{r_{13}^2}{r_{12}^2r_{32}^2}\left(g^2D_2(r_{13})\right). \quad (57)$$

We can consider our new expression as a result of changing the fixed g^2 to a running quantity:

$$g^2\to-\frac{4\pi^3r_{12}^2r_{32}^2}{r_{13}^2}F(r_{12},r_{32})\,. \qquad (58)$$

The function $F(r_1,r_2)$ can be presented in a different form, which demonstrates the absence of an ultraviolet divergency coming from the singular behavior of $\tilde{\eta}(\rho)$ at $\rho\to0$. We have

$$\begin{aligned}&\int \mathrm{d}^2\rho\tilde{\eta}(\rho)\xi^2(r_1-\rho)\\ &=\int \mathrm{d}^2\rho\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q)}{\eta(q_1)\eta(q_2)}\\ &\quad\times\mathrm{e}^{\mathrm{i}q\rho+\mathrm{i}q_1(r_1-\rho)+\mathrm{i}q_2(r_1-\rho)}\\ &=\int\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q_1+q_2)}{\eta(q_1)\eta(q_2)}\mathrm{e}^{\mathrm{i}r_1(q_1+q_2)} \qquad (59)\end{aligned}$$

and similarly

$$\begin{aligned}&\int \mathrm{d}^2\rho\tilde{\eta}(\rho)\xi(r_1-\rho)\xi(r_2-\rho)\\ &=\int \mathrm{d}^2\rho\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q)}{\eta(q_1)\eta(q_2)}\\ &\quad\times\mathrm{e}^{\mathrm{i}q\rho+\mathrm{i}q_1(r_1-\rho)+\mathrm{i}q_2(r_2-\rho)}\\ &=\int\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q_1+q_2)}{\eta(q_1)\eta(q_2)}\mathrm{e}^{\mathrm{i}r_1q_1+\mathrm{i}r_2q_2}\,. \qquad (60)\end{aligned}$$

So we find

$$F(r_1,r_2)=\int\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q_1+q_2)}{\eta(q_1)\eta(q_2)}\left(\mathrm{e}^{\mathrm{i}q_1r_1}-\mathrm{e}^{\mathrm{i}q_1r_2}\right)\times\left(\mathrm{e}^{\mathrm{i}q_2r_1}-\mathrm{e}^{\mathrm{i}q_2r_2}\right). \qquad (61)$$

In this form it is clear that $F(r_1,r_2)$ is a well defined function that does not contain an ultraviolet nor an infrared divergency.

For further use note the identity

$$\begin{aligned}\int \mathrm{d}^2r_1F(r_1-r,r_1)&=\int\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\eta(q_1+q_2)}{\eta(q_1)\eta(q_2)}\\ &\quad\times\left(\mathrm{e}^{-\mathrm{i}q_1r}-1\right)\left(\mathrm{e}^{-\mathrm{i}q_2r}-1\right)\\ &\quad\times\int\mathrm{d}^2r_1e^{ir_1(q_1+q_2)}=0\,, \qquad (62)\end{aligned}$$

since $\eta(0)=0$.

4 Pomeron in the nucleus

With the triple-pomeron vertex known, it is straightforward to obtain the equation that sums fan diagrams describing propagation of the pomeron in the nucleus. Repeating the derivation for the fixed coupling constant in [13] we find for this sum $\Phi(y,b,r)$ at fixed impact parameter b:

$$\begin{aligned}&\Phi(y,r,b)\\ &=\Phi_1(y,r,b)+\frac{1}{2}N_c\int_0^\infty\mathrm{d}y'\prod_{j=1}^3\mathrm{d}^2r_j\delta^2(r_1-r_2+r_3)\\ &\times F(r_2,r_3)\eta^2(-\mathrm{i}\nabla_1)G(y-y',r,r_1)\Phi(y',r_2,b)\Phi(y',r_3,b)\,.\end{aligned} \qquad (63)$$

Here $G(y,r,r')$ is the pomeron forward Green function satisfying the equation

$$\left(\frac{\partial}{\partial y}+H\right)G(y,r,r')=\delta(y)\eta^{-1}(-\mathrm{i}\nabla)\eta^{-1}\times(-\mathrm{i}\nabla')\delta^2(r-r')\,, \qquad (64)$$

with the Hamiltonian H for the non-amputated forward wave function given by

$$H=2\omega+K^\dagger\,, \qquad (65)$$

where ω is expressed via function η according to (1) and

$$K^\dagger(q_1|q_1')=2N_c\frac{\eta(q_1')}{\eta(q_1)\eta(q_1-q_1')}\,. \qquad (66)$$

The inhomogeneous term Φ_1 corresponds to the single pomeron exchange:

$$\Phi_1(y,r,b)=\frac{1}{2}N_cAT(b)\int\mathrm{d}^2r'G(y,r,r')\rho(r')\,, \quad (67)$$

where $T(b)$ is the nuclear profile function, and $\rho(r)$ is the color density of the nucleon. Applying the operator $\partial/\partial y+H$ to (63) we find the evolution equation for the pomeron in

the nucleus:

$$\left(\frac{\partial}{\partial y}+H\right)\Phi(y,r,b)$$
$$=\delta(y)\Phi_0(r,b)+\frac{1}{2}N_c\int\prod_{j=2}^{3}\mathrm{d}^2r_j\delta^2(r-r_2+r_3)F(r_2,r_3)$$
$$\times\Phi(y,r_2,b)\Phi(y,r_3,b)\,, \tag{68}$$

where $\Phi_0(r,b)$, playing the role of the initial condition, is given by

$$\Phi_0(r,b)=\frac{1}{2}AT(b)\eta^{-2}(-\mathrm{i}\nabla)\rho(r)\,. \tag{69}$$

Unlike the case of the fixed coupling constant this equation is not simplified in momentum space.

To compare with the dipole approach, we rewrite our expression (68) as a whole in the coordinate space. To do this we have to rewrite the action of the Hamiltonian (65) on the wave function in coordinate space. In transforming the pomeron amplitude $\Phi(y,q,b)$ to coordinate space we have to take into account the condition

$$\Phi(y,r=0,b)=0\,, \tag{70}$$

which we have extensively used. Technically it means that we have to add to $\Phi(y,q,b)$ a term proportional to $\delta^2(q)$ that guarantees this property. This is essential to obtain the correct form for the linear part of the evolution equation in coordinate space[1].

It is convenient to split (55) into three terms,

$$F(r_1,r_2))=f(r_1,r_1)+f(r_2,r_2)-2f(r_1,r_2)\,, \tag{71}$$

where

$$f(r_1,r_2)=\int\mathrm{d}^2\rho\tilde{\eta}(\rho)\xi(r_1-\rho)\xi(r_2-\rho)\,. \tag{72}$$

In terms of this function one easily finds

$$2\int\frac{\mathrm{d}^2q}{(2\pi)^2}\mathrm{e}^{\mathrm{i}qr}\omega(q)\Phi(y,q,b)=N_c\int\mathrm{d}^2r_1f(r_1-r,r_1-r)$$
$$\times\Phi(y,r_1,b) \tag{73}$$

and

$$\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{\mathrm{d}^2q'}{(2\pi)^2}\mathrm{e}^{\mathrm{i}qr}K^{\dagger}(q|q')\Phi(y,q',b)$$
$$=2N_c\int\mathrm{d}^2r_1f(r_1-r,r_1)\Phi(y,r_1,b)\,. \tag{74}$$

Thus in coordinate space we get

$$H\Phi(y,r,b)=-N_c\int\mathrm{d}^2r_1(2f(r_1-r,r_1)-f(r_1-r,r_1-r))$$
$$\times\Phi(y,r_1)+\mathrm{const}\,, \tag{75}$$

where *const* should be taken to ensure property (70). As a result we find

$$H\Phi(y,r,b)=N_c\int\mathrm{d}^2r_1F(r_1-r,r_1)\Phi(y,r_1,b)$$
$$=\frac{1}{2}N_c\int\mathrm{d}^2r_1F(r_1-r,r_1)(\Phi(y,r_1,b)$$
$$+\Phi(y,r_1-r,b))\,. \tag{76}$$

Using the identity (62) we may add to the bracket any function independent of r_1 to finally obtain

$$H\Phi(y,r,b)=\frac{1}{2}N_c\int\mathrm{d}^2r_1F(r_1-r,r_1)(\Phi(y,r_1,b)$$
$$+\Phi(y,r_1-r,b)-\Phi(y,r,b))\,. \tag{77}$$

In this form the linear part of the evolution equation acquires the standard color dipole structure (see e.g. [2]) and the whole evolution equation becomes

$$\frac{\partial}{\partial y}\Phi(y,r)=-\frac{1}{2}N_c\int\mathrm{d}^2r_1F(r_1-r,r_1)$$
$$\times(\Phi(y,r_1,b)+\Phi(y,r_1-r,b)-\Phi(y,r,b)$$
$$-\Phi(y,r_1,b)\Phi(y,r_1-r,b))\,. \tag{78}$$

In the limit of the fixed coupling constant we find, dropping the infrared regularization terms,

$$f^{\mathrm{fix}}(r_1,r_2)=-\frac{\alpha_{\mathrm{s}}}{\pi^2}\frac{\mathbf{r}_1\mathbf{r}_2}{r_1^2r_2^2}\,, \tag{79}$$

where at $r_1=r_2=r$ one should understand $1/r^2$ as regularized in the ultraviolet [11]:

$$\frac{1}{r^2}\equiv\frac{1}{r^2+\epsilon^2}+2\pi\delta^2(r)\ln\epsilon\,,\quad\epsilon\to0\,. \tag{80}$$

So in analogy with [1] we may define three running coupling constants by

$$f(r_1,r_2)=-\frac{1}{\pi^2}\frac{\alpha_{\mathrm{s}}(r_1)\alpha_{\mathrm{s}}(r_2)}{\alpha_{\mathrm{s}}(r_1,r_2)}\frac{\mathbf{r}_1\mathbf{r}_2}{r_1^2r_2^2}\,, \tag{81}$$

with the additional condition $\alpha_{\mathrm{s}}(r,r)=\alpha_{\mathrm{s}}(r)$, and rewrite (78) as

$$\frac{\partial}{\partial y}\Phi(y,r,b)=\frac{1}{2\pi^2}N_c\int\mathrm{d}^2r_2\,\mathrm{d}^2r_3\delta(r-r_1+r_2)F(r_1,r_2)$$
$$\times\left(\frac{\alpha_{\mathrm{s}}(r_1)}{r_1^2}+\frac{\alpha_{\mathrm{s}}(r_2)}{r_2^2}-2\frac{\alpha_{\mathrm{s}}(r_1)\alpha_{\mathrm{s}}(r_2)}{\alpha_{\mathrm{s}}(r_1,r_2)}\frac{\mathbf{r}_1\mathbf{r}_2}{r_1^2r_2^2}\right)$$
$$\times(\Phi(y,r_1,b)+\Phi(y,r_2,b)-\Phi(y,r,b)$$
$$-\Phi(y,r_1,b)\Phi(y,r_2,b))\,. \tag{82}$$

[1] We are greatly indebted to Y. Kovchegov, who pointed out this circumstance.

5 Discussion

Equations (78) and (82) present our final result for the nonlinear BFKL equation with the running coupling. We stress that the function $\eta(q)$ occurring in them is determined only by its asymptotic form (5) together with the requirement (7). A simple possibility is to choose

$$\eta(q) = \frac{1}{2\pi} bq^2 \ln\left(a + \frac{q^2}{\Lambda^2}\right), \tag{83}$$

with b given by (6) and with arbitrary $a > 1$. Also one has to remember that the equations are derived only in the leading order in the running coupling. Already subleading terms of the relative order $1/\ln(q^2/\Lambda^2)$ remain undetermined, since they correspond to the next-to-leading order in the running coupling.

Our final coordinate space equation (82) fully coincides with (101) in [1] obtained in the dipole formalism (for the forward case). However, in our approach the running couplings $\alpha_s(r)$ and $\alpha_s(r_1, r_2)$ are defined by (81) in a general manner, irrespective of any regularization procedure, and they are determined by the concrete choice of the function $\eta(q)$. In fact, they are fixed only in as far the high-momentum behavior of this function is known and so admit a high degree of arbitrariness. It remains to be seen how this arbitrariness may influence concrete results that follow from the solution of the evolution equation (82).

Our equation for the linear evolution contains the running coupling of different arguments, clearly visible in (1) and (2). It allows for arbitrary relations between the magnitude of the momenta q_i, q_i' and $q_i - q_i'$, $i = 1, 2$. A different method to introduce the running coupling and find the subleading terms has been proposed in [14] based on the study of certain specific regions of these momenta and use of the renormalization group method to improve the equation in the collinear limit. It would be interesting to combine both approaches to incorporate both the bootstrap and the renormalization group into the equation. At present do not have any definite proposals on this point.

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Appendix: Function $F(r_1, r_2)$ with a fixed coupling constant

We check that for a fixed coupling constant the function $F(r_1, r_2)$ indeed passes into the expression corresponding to (57). In the fixed coupling case $\eta(k)$ is given by (4). Then we find

$$\tilde{\eta}(\rho) = -\frac{2\pi}{g^2}\nabla^2\delta^2(\rho) \tag{A.1}$$

and

$$\xi(r) = -\frac{g^2}{(2\pi)^2}(\ln r - c)\,, \tag{A.2}$$

where $c = \ln(2/m) + \psi(1)$ and m is the infrared regularizer (gluon mass). Terms containing c and thus depending on the infrared regularization cancel in the final result. Performing the integration over ρ with the help of the δ-function, we find

$$f^{\rm fix}(r_1, r_2) = -\frac{g^2}{(2\pi)^3}\big((\ln r_1 - c)\nabla_2^2 \ln r_2 + (\ln r_2 - c) \times \nabla_2^2 \ln r_1 + 2\nabla_1 \ln r_1 \nabla_2 \ln r_2\big). \tag{A.3}$$

Taking into account that

$$\nabla^2 \ln r = 2\pi\delta^2(r)\,, \tag{A.4}$$

we find

$$f^{\rm fix} = -\frac{g^2}{8\pi^3}\left(2\frac{\mathbf{r}_1\mathbf{r}_2}{r_1^2 r_2^2} + 2\pi\delta^2(r_1)(\ln r_2 - c) + 2\pi\delta^2(r_2)(\ln r_1 - c)\right). \tag{A.5}$$

Forming $F(r_1, r_2)$ according to (71) we see that terms with c cancel, and dropping terms proportional to $\delta^2(r_1)$ or $\delta^2(r_2)$ we find

$$F^{\rm fix}(r_1, r_2) = -\frac{g^2}{4\pi^3}\frac{(r_1 - r_2)^2}{r_1^2 r_2^2}\,, \tag{A.6}$$

in full correspondence with (57).

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